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APPLICATION OF THE ABSOLUTE NODAL COORDINATE FORMULATION TO MULTIBODY SYSTEM DYNAMICS

Department of Mechanical Engineering

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APPLICATION OF THE ABSOLUTE NODAL COORDINATE FORMULATION TO MULTIBODY SYSTEM DYNAMICS

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ABSTRACT

The floating frame of reference formulation is currently the most widely used approach in flexible multibody simulations. The use of this approach, however, has been limited to small deformation problems. In this investigation, the use of the new *absolute nodal coordinate formulation* in the small and large deformation analysis of flexible multibody systems that consist of interconnected bodies is discussed. While in the floating frame of reference formulation a mixed set of absolute reference and local elastic coordinates are used, in the absolute nodal coordinate formulation only absolute coordinates are used. In the absolute nodal coordinate formulation, new interpretation of the nodal coordinates of the finite elements is used. No infinitesimal or finite rotations are used as nodal coordinates for beams and plates, instead global slopes are used to define the element nodal coordinates. Using this interpretation of the element coordinates beams and plates can be considered as isoparametric elements, and as a result, exact modeling of the rigid body dynamics can be obtained using the element shape function and the absolute nodal coordinates. Unlike the floating frame of reference approach, no coordinate transformation is required in order to determine the element inertia. The mass matrix of the finite elements is a constant matrix, and therefore, the centrifugal and Coriolis forces are equal to zero when the absolute nodal coordinate formulation is used. The generalized elastic forces, however, become highly nonlinear function of the system coordinates, and as such, little is to be gained from the use of the small strain assumptions. Another advantage of using the absolute nodal coordinate formulation in the dynamic simulation of multibody systems is its simplicity in imposing some of the joint constraints and also its simplicity in formulating the generalized forces due to spring-damper elements. The results obtained in this investigation shows an excellent agreement with the results obtained using the floating frame of reference formulation when large rotation-small deformation problems are considered.

1. INTRODUCTION

The formulation of the equations of motion of flexible multibody systems using the finite element method has been a challenging problem, particularly when conventional non-isoparametric elements such as beams and plates are used. The nodal coordinates of these widely use elements include infinitesimal rotations. As a result, exact modeling of the rigid body dynamics can not be obtained when these non-isoparametric elements are used [6]. Such a limitation poses a serious problem when flexible multibody dynamics are considered. Generally these systems consist of interconnected rigid and deformable bodies, each of which may undergo large rotations. For this reason, several formulations that lead to exact modeling of the rigid body inertia were proposed for the nonlinear dynamic analysis of flexible multibody systems. Among these formulations is the floating frame of reference approach [1-5] which can be used to obtain accurate modeling of the rigid body dynamics and also leads to zero strain under an arbitrary rigid body motion of the non-isoparametric finite elements. The floating frame of reference approach uses two sets of coordinates to describe the dynamics of deformable bodies that undergo large reference displacements. The large reference translations and rotations are described by a mixed set of absolute Cartesian and orientation coordinates defined in a global inertial frame of reference. The elastic displacements of the bodies are defined with respect to its coordinate system using the nodal coordinates of the finite elements. The body frame of reference is defined using an appropriate set of reference conditions that define an unique displacement field [5]. The equations of motion obtained using the floating frame of reference formulation exhibit a strong nonlinear inertia coupling

between the reference and elastic coordinates. The mass matrix is highly nonlinear and the inertia forces include Coriolis and centrifugal forces, which are quadratic in the velocities. The stiffness matrix, on the other hand, takes a simple form and it is the same as the stiffness matrix that appears in structural mechanics.

The use of two different types of frames of reference; global and local (inertial and non-inertial), to describe two different sets of coordinates (reference coordinates and elastic coordinates), leads to the complexity of the resulting inertia forces. If isoparametric finite elements, which have absolute nodal coordinates defined in the inertial frame of reference, are used to model the flexible bodies; much simpler expressions for the inertia forces can be obtained. Furthermore, the shape function and the nodal coordinates of the element can be used to obtain exact modeling of the rigid body dynamics provided that the finite element shape functions have a complete set of rigid body modes. In the absolute nodal coordinate formulation [7-8], a new interpretation of the nodal coordinates is used in order to develop new isoparametric beam and plate elements. Unlike the work of Simo and Vu-Quoc [9], no finite rotations are used as nodal coordinates, and instead global slopes are used as nodal coordinates. The use of finite rotations as nodal coordinates can lead to redundancy in representing the large rotation of the cross section of the finite element[7].

In addition to the fact that the absolute nodal coordinate formulation automatically captures the nonlinear effects arising from the coupling between different modes of displacements, the formulation of the joint constraints and forces becomes simpler when this new approach is used in flexible multibody dynamics. It is the objective of this

investigation to examine and demonstrate the use of this new finite element procedure in the small and large deformation analysis of flexible multibody systems that consist of interconnected bodies. Comparison will be made with the floating frame of reference formulation, which is currently the most widely used computer procedure for the analysis of flexible multibody systems. Throughout the analysis presented in this paper, a two dimensional beam element is used for demonstration purposes.

This paper is organized as follows. In Section 2, the absolute nodal coordinate formulation is presented, and the constant element mass matrix and nonlinear stiffness matrix are identified. In Section 3 the formulation of the generalized forces, when the absolute nodal coordinate formulation is used, is presented. It is shown in this section some of the fundamental differences between the absolute nodal coordinate formulation and other existing finite element procedures. Because of the use of global slopes as element nodal coordinates, new set of generalized moments must be used. In section 4, the formulation of the joint constraints in the absolute nodal coordinate formulation is discussed. Sections 5-7 demonstrate the equivalence of the absolute nodal coordinate formulation and the floating frame of reference approach. Examples are presented in Section 8 and the numerical results obtained using the absolute nodal coordinate formulation are compared with the results obtained using the floating frame of reference approach. Summary and conclusions drawn from the analysis developed in this paper are presented in Section 9.

2. ABSOLUTE NODAL COORDINATE FORMULATION

In the mixed finite element formulations, displacements and displacement gradients are used as nodal coordinates. These conventional finite element mixed formulations, however, have serious limitations when flexible multibody applications are considered. For instance, most of the mixed formulations were used in the framework of incremental procedure and the shape functions employed often do not have a complete set of rigid body modes. Furthermore, in structural dynamics applications mixed formulations are often used with lumped masses. When a lumped mass formulation is used with conventional beam elements, exact modeling of rigid body dynamics cannot be obtained. In the absolute nodal coordinate formulation used in this investigation, it is required that the element shape function has a complete set of rigid body modes that can describe arbitrary rigid body translational and rotational displacements. Global displacements and slopes are used as nodal coordinates. By so doing, exact modeling of the rigid body dynamics can be obtained when only a consistent mass formulation is used.

In the absolute nodal coordinate formulation, the coordinates of the material points are defined in the global system. These absolute coordinates as shown in Fig. 1 are defined in terms of the element shape function and the vector of nodal coordinates as

$$\mathbf{r} = \begin{bmatrix} r_x \\ r_y \end{bmatrix} = \mathbf{S}\mathbf{e} \tag{1}$$

where $\mathbf{r}$ is the global position vector of an arbitrary point on the element, $\mathbf{S}$ is a global shape function that include a complete set of rigid body modes, and $\mathbf{e}$ is the vector of nodal coordinates that includes global displacements and slopes defined at the nodal points of the element.

2.1. Displacement Field and Rigid Body Kinematics

In this paper, a planar beam element is used as an example to demonstrate the use of the finite element absolute nodal coordinate formulation in flexible multibody applications. Since the coordinates of the material points in this formulation are defined in a global frame of reference, there is no reason to use different polynomials to interpolate the displacement components. In this investigation, a cubic polynomial is used for both components of the displacement. In this case, the element shape function and the vector of nodal coordinates are defined as

$$\mathbf{S}=\begin{bmatrix} 1-3\xi^2+2\xi^3 & 0 & l(\xi-2\xi^2+\xi^3) & 0 & 3\xi^2-2\xi^3 & 0 & l(\xi^3-\xi^2) & 0 \\ 0 & 1-3\xi^2+2\xi^3 & 0 & l(\xi-2\xi^2+\xi^3) & 0 & 3\xi^2-2\xi^3 & 0 & l(\xi^3-\xi^2) \end{bmatrix} \quad (2)$$

$$\mathbf{e}=\begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{bmatrix} \quad (3)$$

where the elements of the vector of nodal coordinates are defined as

$$\begin{aligned} &e_1=r_x(x=0), \quad e_2=r_y(x=0), \quad e_3=\frac{\partial r_x(x=0)}{\partial x}, \quad e_4=\frac{\partial r_y(x=0)}{\partial x}, \\ &e_5=r_x(x=l), \quad e_6=r_y(x=l); \quad e_7=\frac{\partial r_x(x=l)}{\partial x}, \quad e_8=\frac{\partial r_y(x=l)}{\partial x} \end{aligned} \quad (4)$$

where x is the spatial coordinate along the element axis. Note that in the absolute nodal coordinate formulation no infinitesimal rotations are used as nodal coordinates, instead, slopes are used. The initial values of the global slopes in the undeformed reference configuration can be determined using simple rigid body kinematics by utilizing the fact that Eq. 1 can be used to obtain exact modeling of the kinematics of rigid bodies. For instance, in an arbitrary undeformed reference configuration defined by the translations $r_x(x = 0)$ and $r_y(x = 0)$ and the rigid body rotation θ the global position of an arbitrary point on the beam can be written as

$$\mathbf{r}(x) = \begin{bmatrix} r_x(x) \\ r_y(x) \end{bmatrix} = \mathbf{Se} = \begin{bmatrix} r_x(x=0) + x\cos\theta \\ r_y(x=0) + x\sin\theta \end{bmatrix} \tag{5}$$

It follows that the global slopes in the undeformed reference configuration are defined as

$$e_3 = e_7 = \cos\theta; \qquad e_4 = e_8 = \sin\theta \tag{6}$$

A similar procedure can be used to determine the global slopes in the case of three dimensional elements.

2.2. Kinetic Energy

The kinetic energy of the beam element is defined as

$$T = \frac{1}{2}\int_V \rho \dot{\mathbf{r}}^T \dot{\mathbf{r}} dV = \frac{1}{2}\dot{\mathbf{e}}^T \left(\int_V \rho \mathbf{S}^T \mathbf{S} dV\right)\dot{\mathbf{e}} = \frac{1}{2}\dot{\mathbf{e}}^T \mathbf{M}_a \dot{\mathbf{e}} \tag{7}$$

where V is the volume, ρ is the mass density of the beam material, and $\mathbf{M}_a$ is the mass matrix of the element. Note that the mass matrix in Eq. 7 is symmetric and constant, and it is the same matrix that appears in linear structural dynamics. Using the shape function of Eq. 2, the mass matrix of the element can be evaluated as

$$\mathbf{M}_a = \int_V \rho \mathbf{S}^T \mathbf{S} dV = m \begin{bmatrix} \frac{13}{35} & 0 & \frac{11l}{210} & 0 & \frac{9}{70} & 0 & -\frac{13l}{420} & 0 \\ & \frac{13}{35} & 0 & \frac{11l}{210} & 0 & \frac{9}{70} & 0 & -\frac{13l}{420} \\ & & \frac{l^2}{105} & 0 & \frac{13l}{420} & 0 & -\frac{l^2}{140} & 0 \\ & & & \frac{l^2}{105} & 0 & \frac{13l}{420} & 0 & -\frac{l^2}{140} \\ & & & & \frac{13}{35} & 0 & -\frac{11l}{210} & 0 \\ & \mathit{Symmetric} & & & & \frac{13}{35} & 0 & -\frac{11l}{210} \\ & & & & & & \frac{l^2}{105} & 0 \\ & & & & & & & \frac{l^2}{105} \end{bmatrix} \tag{8}$$

where m is the mass of the beam and l is its length. It can be demonstrated that the use of this mass matrix leads to exact modeling of the rigid body inertia [8].

2.3 Strain Energy

While the absolute nodal coordinate formulation leads to a simple expression for the inertia forces, the use of this formulation results in a relatively complex expression for the elastic forces. In order to demonstrate this fact, a simple linear elastic model based on the classical beam theory is used in this section. If point O shown in Fig. 1 is used as the reference point, the displacements of an arbitrary point on the beam relative to point O may be written as

$$\mathbf{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} (\mathbf{S}_1 - \mathbf{S}_{1O})\mathbf{e} \\ (\mathbf{S}_2 - \mathbf{S}_{2O})\mathbf{e} \end{bmatrix} \tag{9}$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ are the rows of the element shape function matrix, and $\mathbf{S}_{1O}$ and $\mathbf{S}_{2O}$ are the rows of the element shape function matrix defined at point O. In order to define these relative displacements in the element coordinate system , two unit vectors $\mathbf{i}$ and $\mathbf{j}$ along the element axis are defined as

$$\mathbf{i} = \begin{bmatrix} i_x \\ i_y \end{bmatrix} = \frac{\mathbf{r}_A - \mathbf{r}_O}{|\mathbf{r}_A - \mathbf{r}_O|}, \quad \mathbf{j} = \begin{bmatrix} j_x \\ j_y \end{bmatrix} = \mathbf{k} \times \mathbf{i} \tag{10}$$

where $\mathbf{k}$ is an unit vector along the Z axis. The longitudinal and transverse deformations of the beam can then be defined as

$$\mathbf{u}_d = \begin{bmatrix} u_l \\ u_t \end{bmatrix} = \begin{bmatrix} \mathbf{u}^T \mathbf{i} - x \\ \mathbf{u}^T \mathbf{j} \end{bmatrix} = \begin{bmatrix} u_x i_x + u_y i_y - x \\ u_x j_x + u_y j_y \end{bmatrix} \tag{11}$$

The strain energy of the beam element due to the longitudinal and transverse displacements is given by:

$$U = \frac{1}{2} \int_0^l \left(Ea\left(\frac{\partial u_l}{\partial x}\right)^2 + EI\left(\frac{\partial^2 u_t}{\partial x^2}\right)^2 \right) dx = \frac{1}{2} \mathbf{e}^T \mathbf{K}_a \mathbf{e} \tag{12}$$

where E is the modulus of elasticity, a is the cross sectional area, I is the second moment of area of the beam element, and $\mathbf{K}_a$ is the element stiffness matrix. This matrix is a nonlinear function of time. It can be shown that the strain energy can be expressed in terms of the following *stiffness shape integrals*:

$$\mathbf{A}_{11} = \frac{Ea}{l}\int_0^1 \left[\left(\frac{\partial \mathbf{S}_1}{\partial \xi}\right)^T \left(\frac{\partial \mathbf{S}_1}{\partial \xi}\right)\right] d\xi, \quad \mathbf{A}_{12} = \frac{Ea}{l}\int_0^1 \left[\left(\frac{\partial \mathbf{S}_1}{\partial \xi}\right)^T \left(\frac{\partial \mathbf{S}_2}{\partial \xi}\right)\right] d\xi$$

$$\mathbf{A}_{21} = \frac{Ea}{l}\int_0^1 \left[\left(\frac{\partial \mathbf{S}_2}{\partial \xi}\right)^T \left(\frac{\partial \mathbf{S}_1}{\partial \xi}\right)\right] d\xi, \quad \mathbf{A}_{22} = \frac{Ea}{l}\int_0^1 \left[\left(\frac{\partial \mathbf{S}_2}{\partial \xi}\right)^T \left(\frac{\partial \mathbf{S}_2}{\partial \xi}\right)\right] d\xi$$

$$\mathbf{B}_{11} = \frac{EI}{l^3}\int_0^1 \left[\left(\frac{\partial^2 \mathbf{S}_1}{\partial \xi^2}\right)^T \left(\frac{\partial^2 \mathbf{S}_1}{\partial \xi^2}\right)\right] d\xi, \quad \mathbf{B}_{12} = \frac{EI}{l^3}\int_0^1 \left[\left(\frac{\partial^2 \mathbf{S}_1}{\partial \xi^2}\right)^T \left(\frac{\partial^2 \mathbf{S}_2}{\partial \xi^2}\right)\right] d\xi \qquad (13)$$

$$\mathbf{B}_{21} = \frac{EI}{l^3}\int_0^1 \left[\left(\frac{\partial^2 \mathbf{S}_2}{\partial \xi^2}\right)^T \left(\frac{\partial^2 \mathbf{S}_1}{\partial \xi^2}\right)\right] d\xi, \quad \mathbf{B}_{22} = \frac{EI}{l^3}\int_0^1 \left[\left(\frac{\partial^2 \mathbf{S}_2}{\partial \xi^2}\right)^T \left(\frac{\partial^2 \mathbf{S}_2}{\partial \xi^2}\right)\right] d\xi$$

$$\mathbf{A}_1 = Ea\int_0^1 \left(\frac{\partial \mathbf{S}_1}{\partial \xi}\right)^T d\xi, \qquad \mathbf{A}_2 = Ea\int_0^1 \left(\frac{\partial \mathbf{S}_2}{\partial \xi}\right)^T d\xi$$

Where the explicit forms of these matrices obtained using the shape function of Eq. 2 are given in the appendix of the paper. Using these stiffness shape integrals, the generalized elastic forces of the element can be calculated from

$$\left(\frac{\partial U}{\partial \mathbf{e}}\right)^T = \mathbf{A}_{11}\,\mathbf{e}\,i_x^{\,2} + \mathbf{A}_{22}\,\mathbf{e}\,i_y^{\,2} + (\mathbf{A}_{12}+\mathbf{A}_{21})\,\mathbf{e}\,i_x\,i_y - \mathbf{A}_1\,i_x - \mathbf{A}_2\,i_y + \mathbf{B}_{11}\,\mathbf{e}\,j_x^{\,2} + \mathbf{B}_{22}\,\mathbf{e}\,j_y^{\,2} + (\mathbf{B}_{12}+\mathbf{B}_{21})\,\mathbf{e}\,j_x\,j_y$$
$$+\left(\mathbf{e}^T\mathbf{A}_{11}\mathbf{e}\,i_x + \frac{1}{2}\mathbf{e}^T(\mathbf{A}_{12}+\mathbf{A}_{21})\,\mathbf{e}\,i_y - \mathbf{A}_1^{\,T}\mathbf{e}\right)\left(\frac{\partial i_x}{\partial \mathbf{e}}\right)^T + \left(\mathbf{e}^T\mathbf{A}_{22}\,\mathbf{e}\,i_y + \frac{1}{2}\mathbf{e}^T(\mathbf{A}_{12}+\mathbf{A}_{21})\,\mathbf{e}\,i_x - \mathbf{A}_2^{\,T}\mathbf{e}\right)\left(\frac{\partial i_y}{\partial \mathbf{e}}\right)^T$$
$$+\left(\mathbf{e}^T\mathbf{B}_{11}\,\mathbf{e}\,j_x + \frac{1}{2}\mathbf{e}^T(\mathbf{B}_{12}+\mathbf{B}_{21})\,\mathbf{e}\,j_y\right)\left(\frac{\partial j_x}{\partial \mathbf{e}}\right)^T + \left(\mathbf{e}^T\mathbf{B}_{22}\,\mathbf{e}\,j_y + \frac{1}{2}\mathbf{e}^T(\mathbf{B}_{12}+\mathbf{B}_{21})\,\mathbf{e}\,j_x\right)\left(\frac{\partial j_y}{\partial \mathbf{e}}\right)^T \qquad (14)$$

Where

$$\left(\frac{\partial i_x}{\partial \mathbf{e}}\right)^T = \left(\frac{\partial i_y}{\partial \mathbf{e}}\right)^T = D\left[-(e_6-e_2)^2 \quad (e_5-e_1)(e_6-e_2) \quad 0 \quad 0 \quad (e_6-e_2)^2 \quad -(e_5-e_1)(e_6-e_2) \quad 0 \quad 0\right]^T$$
$$\left(\frac{\partial i_y}{\partial \mathbf{e}}\right)^T = -\left(\frac{\partial j_x}{\partial \mathbf{e}}\right)^T = D\left[(e_5-e_1)(e_6-e_2) \quad -(e_5-e_1)^2 \quad 0 \quad 0 \quad -(e_5-e_1)(e_6-e_2) \quad (e_5-e_1)^2 \quad 0 \quad 0\right]^T \qquad (15)$$
$$D = \frac{1}{\left((e_5-e_1)^2 + (e_6-e_2)^2\right)^{3/2}}$$

2.4. Equations of Motion

Using the principle of virtual work in dynamics and the expression of the kinetic and strain energies given by Eq. 7 and Eq. 12, the equation of motion of the finite element can be written as

$$\mathbf{M}_a \ddot{\mathbf{e}} = \mathbf{Q} \tag{16}$$

Where **Q** is the vector of generalized external nodal forces including the elastic forces. Note that centrifugal and Coriolis forces are equal to zero since the mass matrix is constant. The equations of motion of the deformable body can be obtained by assembling the equations of its elements using a standard finite element procedure.

3. FORMULATION OF THE GENERALIZED EXTERNAL FORCES

It is clear from the analysis presented in the preceding section that there are several fundamental differences between the absolute nodal coordinate formulation and some of the existing finite element procedures. One of these differences is the fact that there is no need to use coordinate transformation in order to determine the element mass matrix. Another difference is attributed to the formulation of the stiffness matrix, which is highly nonlinear in the case of the absolute nodal coordinate formulation even in the case of simple linear elastic model. For this reason, little is to be gained from the use of small deformation assumptions.

Another fundamental difference is due to the nature of the coordinates used in the absolute nodal coordinate formulation. These coordinates do not include infinitesimal or finite rotations. As such, attention must be paid to the definition of the generalized forces associated with the global slopes of the finite element. In this section, the definition of the generalized forces in the absolute nodal coordinate formulation is discussed.

3.1. Force Vector

The virtual work due to an externally applied force **F** acting on an arbitrary point on the element is given by $\mathbf{F}^T\delta\mathbf{r}$, where **r** is the position vector of the point of application of the force and $\delta\mathbf{r}$ is the virtual change in the vector **r**. In order to obtain the generalized forces associated with the absolute nodal coordinates it is necessary to express $\delta\mathbf{r}$ in terms of the virtual displacements of these nodal coordinates. To this end, one can write

$$\mathbf{F}^T\delta\mathbf{r} = \mathbf{F}^T\mathbf{S}\delta\mathbf{e} = \mathbf{Q}_F{}^T\delta\mathbf{e} \tag{17}$$

Where $\mathbf{Q}_F = \mathbf{S}^T\mathbf{F}$ is the vector of generalized forces associated with the element nodal coordinates. For example, the virtual work due to the distributed gravity of the finite element can be obtained using the shape function of Eq. 2 as

$$\int_0^l \begin{bmatrix} 0 & -\rho g \end{bmatrix} \mathbf{S}\delta\mathbf{e} \; dV = mg\begin{bmatrix} 0 & -\dfrac{1}{2} & 0 & -\dfrac{l}{12} & 0 & -\dfrac{1}{2} & 0 & \dfrac{l}{12} \end{bmatrix}\delta\mathbf{e} \tag{18}$$

Which defines the vector of generalized distributed gravity forces as

$$\mathbf{Q}_F = mg\begin{bmatrix} 0 & -\dfrac{1}{2} & 0 & -\dfrac{l}{12} & 0 & -\dfrac{1}{2} & 0 & \dfrac{l}{12} \end{bmatrix}^T \tag{19}$$

3.2. Moment

When a moment M acts at a cross section of the beam, the virtual work due to this moment is given by $M\delta\alpha$, where α is the angle of rotation of the cross section. The orientation of a coordinate system whose origin is rigidly attached to this cross section (see Fig. 1) can be defined using the following transformation matrix:

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \frac{1}{d^{\frac{1}{2}}} \begin{bmatrix} \dfrac{\partial r_x}{\partial x} & -\dfrac{\partial r_y}{\partial x} \\ \dfrac{\partial r_y}{\partial x} & \dfrac{\partial r_x}{\partial x} \end{bmatrix} \qquad (20)$$

$$d = \left(\frac{\partial r_x}{\partial x}\right)^2 + \left(\frac{\partial r_y}{\partial x}\right)^2$$

Using the elements of the planar transformation matrix given in the preceding equation, one has

$$\sin\alpha = d^{-\frac{1}{2}}\left(\frac{\partial r_y}{\partial x}\right), \qquad \cos\alpha = d^{-\frac{1}{2}}\left(\frac{\partial r_x}{\partial x}\right) \qquad (21)$$

Using these two equations, it can be shown that

$$\delta\alpha = \frac{\dfrac{\partial r_x}{\partial x}\delta\left(\dfrac{\partial r_y}{\partial x}\right) - \dfrac{\partial r_y}{\partial x}\delta\left(\dfrac{\partial r_x}{\partial x}\right)}{d} \qquad (22)$$

If the concentrated moment M is applied, for example, at node O of the element, the generalized forces due to this moment are defined as

$$\mathbf{Q}_M = \begin{bmatrix} 0 & 0 & \dfrac{-Me_4}{d} & \dfrac{Me_3}{d} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{23}$$

3.3. Spring-Damper Forces

The formulation of the generalized forces due to a spring-damper element connecting two finite elements is very simple as compared to the floating frame of reference formulation which leads to a highly nonlinear complex expression for these forces [5]. In the absolute nodal coordinate formulation, the generalized forces due to a spring-damper element take a simple form due to the fact that absolute coordinates are used. If a and b are the nodes to which the ends of the spring-damper element are attached, the generalized forces acting at node b simply take the form

$$\mathbf{Q}_{SD} = k\begin{bmatrix} e_1{}^a - e_1{}^b \\ e_2{}^a - e_2{}^b \end{bmatrix} + c\begin{bmatrix} \dot{e}_1{}^a - \dot{e}_1{}^b \\ \dot{e}_2{}^a - \dot{e}_2{}^b \end{bmatrix} \tag{24}$$

Where k and c are the spring and damping constants, respectively.

4. FORMULATION OF CONSTRAINTS

The formulation of many of the constraints equations that describe mechanical joints in flexible multibody systems become relatively simple when the absolute nodal coordinate formulation is used. In many cases, these constraint equations take a complex nonlinear form when the floating frame of reference approach is used. This mainly due to the fact that in the floating frame of reference formulation, two sets of coordinates

(reference and elastic) defined in two different frames of reference (global and body) are used. In the absolute nodal coordinate formulation, only one set of absolute coordinates defined in one global coordinate system is used. As a consequence, many of the constraint equations become simple and linear. For instance, the revolute joint constraints, which are highly nonlinear in the floating frame of reference formulation [5], become simple and linear when the absolute nodal coordinate formulation is used. Figure 2 shows two elements i and j, which are connected, by a revolute joint at point P. The constraint equation for the revolute joint can be written as

$$\mathbf{r}_P{}^i = \mathbf{r}_P{}^j \tag{25}$$

Which can be written in terms of the element coordinates as

$$\mathbf{S}_P^i \mathbf{e}^i = \mathbf{S}_P^j \mathbf{e}^j \tag{26}$$

Where $\mathbf{S}_P^i$ and $\mathbf{S}_P^j$ are the shape functions of the elements i and j evaluated at point P, and $\mathbf{e}^i$ and $\mathbf{e}^j$ are the vectors of nodal coordinates of the two elements. If point P is selected as a nodal point on the two elements, the constraint equation of the revolute joint reduce to

$$\begin{bmatrix} e_5^i - e_1^j \\ e_6^i - e_2^j \end{bmatrix} = \mathbf{0} \tag{27}$$

Where e_5^i and e_6^i are the absolute translational nodal coordinates of element i at node P, and e_1^j and e_2^j are the absolute translational nodal coordinates of element j at node P.

5. COMPARISON WITH THE FLOATING FRAME OF REFERENCE FORMULATION

In the floating frame of reference formulation, not all coordinates represent absolute variables, since the configuration of the body is described using a mixed set of absolute reference and local deformation coordinates. The reference coordinates define the location and the orientation of a selected body coordinate system. The deformation of the body is described using a set of local shape functions and a set of deformation coordinates defined in the body coordinate system. In the floating frame of reference formulation, it is assumed that there is no rigid body motion between the body and its coordinate system. Using Fig. 1 and the reference and deformation coordinates, the global position vector of an arbitrary point on the centerline of the beam element can be written as [5]

$$\mathbf{r} = \mathbf{R} + \mathbf{A}\mathbf{u} \tag{28}$$

Where $\mathbf{R} = \mathbf{R}(t)$ defines the global position vector of the origin of the selected beam coordinate system, $\mathbf{A} = \mathbf{A}(t)$ is the transformation matrix that defines the orientation of the selected beam coordinate system with respect to the inertial frame, and $\mathbf{u} = \mathbf{u}(x,t)$ is the local position vector of the arbitrary point defined with respect to the origin of the beam coordinate system. The local position vector $\mathbf{u}$ may be represented in terms of local shape functions $\mathbf{S}_l(x)$ as

$$\mathbf{u}(x,t) = \mathbf{S}_l(x)\mathbf{q}_f(t) \tag{29}$$

where $\mathbf{q}_f(t)$ is the vector of time dependent deformation coordinates which can also be used in the finite element formulation to interpolate the local position as well as the deformation. When the kinematic description of Eq. 28 is used, it is assumed that there is no rigid body motion between the beam and its coordinate system. As a consequence, it is required that the local shape function matrix $\mathbf{S}_l(x)$ contains no rigid body modes. Using Eq. 28 and 29, the motion of the flexible beam can be described using the floating frame of reference formulation as

$$\mathbf{r} = \mathbf{R} + \mathbf{A}\mathbf{S}_l\mathbf{q}_f \tag{30}$$

where the vector $\mathbf{q}_f(t)$ describes the local position and the deformation of an arbitrary point [5], and the vector

$$\mathbf{q}_r(t) = \begin{bmatrix} \mathbf{R}(t) \\ \theta(t) \end{bmatrix} \tag{31}$$

describes the reference motion. In Eq. 31, θ is the angle that defines the orientation of the beam coordinate system. Therefore, the vector of generalized coordinates of the beam used in the floating frame of reference formulation can be written in a partitioned form as

$$\mathbf{q} = \begin{bmatrix} \mathbf{R}^T & \theta & \mathbf{q}_f^T \end{bmatrix}^T = \begin{bmatrix} \mathbf{q}_r^T & \mathbf{q}_f^T \end{bmatrix}^T \tag{32}$$

Using Eq. 30 and the coordinate partitioning of Eq. 32, it can be shown that the mass matrix of the deformable beam in the case of the floating frame of reference formulation can be written in a partitioned form as [5]

$$\mathbf{M}_f = \begin{bmatrix} \mathbf{m}_{rr} & \mathbf{m}_{rf} \\ \mathbf{m}_{fr} & \mathbf{m}_{ff} \end{bmatrix} \tag{33}$$

Unlike the absolute nodal coordinate formulation which leads to a simple mass matrix, the mass matrix in the preceding equation is highly nonlinear in the coordinates $\mathbf{q} = \begin{bmatrix} \mathbf{q}_r^T & \mathbf{q}_f^T \end{bmatrix}^T$ as the result of the dynamic coupling between the reference coordinates $\mathbf{q}_r$ and the deformation coordinates $\mathbf{q}_f$. In the case of planar motion, one has

$$\mathbf{q}_r = \begin{bmatrix} R_x & R_y & \theta \end{bmatrix}^T, \quad \mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{34}$$

In this case of planar motion, it can be shown that the nonlinear mass matrix and the Coriolis and centrifugal forces of the finite element can be expressed in terms of the following constant inertia shape integrals [5]:

$$\overline{\mathbf{S}} = \int_V \rho \mathbf{S}_l dV, \quad \mathbf{m}_{ff} = \int_V \rho \mathbf{S}_l^T \mathbf{S}_l dV, \quad \widetilde{\mathbf{S}} = \int_V \rho \mathbf{S}_l^T \widetilde{\mathbf{I}} \mathbf{S}_l dV \tag{35}$$

Where ρ and V are the mass density and volume of the element, and

$$\widetilde{\mathbf{I}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{36}$$

By establishing the relationship between the coordinates used in the floating frame of reference formulation and the coordinates used in the absolute nodal coordinate formulation, the nonlinear mass matrix of Eq. 33 can be obtained using the constant mass matrix of Eq. 8.

6. RELATIONSHIP BETWEEN THE COORDINATES

In the absolute nodal coordinate formulation, beams and plates can be considered as isoparametric elements. Using this fact, the equivalence between the floating frame of reference formulation and the absolute nodal coordinate formulation can be demonstrated and used to examine the effect of using the consistent mass distribution on the inertia representation of deformable bodies that undergo large reference displacements. In order to demonstrate the equivalence of the floating frame of reference formulation and the absolute nodal coordinate formulation, the relationship between the absolute and local slopes is first defined and then used to establish the relationship between the coordinates used in the two different formulations. In this paper, as an example, the cubic polynomials will be used to equally represent the displacement components of the beam element. The procedure developed in this paper, however, can be applied to other interpolating functions, provided that the global shape function has a complete set of rigid body modes.

6.1. Slope Relationship

Using Eq. 28, the global position vector of an arbitrary point on the beam element can be written using the floating frame of reference formulation as

$$\mathbf{r}(x,t)=\begin{bmatrix} r_x \\ r_y \end{bmatrix}=\begin{bmatrix} R_x+u_x\cos\theta-u_y\sin\theta \\ R_y+u_x\sin\theta+u_y\cos\theta \end{bmatrix} \tag{37}$$

where u_x and u_y are the position coordinates of the arbitrary point defined with respect to the beam coordinate system. It follows in the case of a slender beam element that

$$\left.\begin{aligned}\frac{\partial r_x}{\partial x} &= \frac{\partial u_x}{\partial x}\cos\theta - \frac{\partial u_y}{\partial x}\sin\theta \\ \frac{\partial r_y}{\partial x} &= \frac{\partial u_x}{\partial x}\sin\theta + \frac{\partial u_y}{\partial x}\cos\theta\end{aligned}\right\} \tag{38}$$

This slope relationship plays a fundamental role in defining the relationship between the coordinates used in the absolute nodal coordinate formulation and the coordinates used in the floating frame of reference formulation.

6.2. Coordinate Transformation

In the remainder of this section, we develop the relationship between the coordinates used in the floating frame of reference formulation and the coordinates used in the absolute nodal coordinate formulation. In the case of the absolute nodal coordinate formulation, we use the global element shape function defined by Eq. 2. In the floating frame of reference formulation, we assume that the origin of the beam coordinate system is located at point O and one of the axes connects points O and A. In this case, the local shape function can be obtained from the global shape function of Eq. 2 as

$$\mathbf{S}_l = \begin{bmatrix} l\left(\xi - 2\xi^2 + \xi^3\right) & 0 & 3\xi^2 - 2\xi^3 & l\left(\xi^3 - \xi^2\right) & 0 \\ 0 & l\left(\xi - 2\xi^2 + \xi^3\right) & 0 & 0 & l\left(\xi^3 - \xi^2\right) \end{bmatrix} \tag{39}$$

Note that this local shape function does not include any rigid body modes. The vector $\mathbf{q}_f$ in this case can be defined as

$$\mathbf{q}_f = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 \end{bmatrix}^T \tag{40}$$

Where q_3 is the local x coordinate of the node at A defined in the beam coordinate system, and

$$q_1 = \frac{\partial u_x(x=0)}{\partial x}, \quad q_2 = \frac{\partial u_y(x=0)}{\partial x} \quad q_4 = \frac{\partial u_x(x=l)}{\partial x}, \quad q_5 = \frac{\partial u_y(x=l)}{\partial x} \tag{41}$$

The vector **e** of Eq. 3 used in the absolute nodal coordinate formulation can be expressed in this case in terms of the component of the vector

$$\mathbf{q} = \begin{bmatrix} R_x & R_y & \theta & q_1 & q_2 & q_3 & q_4 & q_5 \end{bmatrix}^T \tag{42}$$

of the floating frame of reference formulation using Eq. 38 as

$$\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \end{bmatrix} = \begin{bmatrix} R_x \\ R_y \\ q_1 \cos\theta - q_2 \sin\theta \\ q_1 \sin\theta + q_2 \cos\theta \\ R_x + q_3 \cos\theta \\ R_y + q_3 \sin\theta \\ q_4 \cos\theta - q_5 \sin\theta \\ q_4 \sin\theta + q_5 \cos\theta \end{bmatrix} \tag{43}$$

Using this vector, it can be shown that

$$\mathbf{Se} = \mathbf{R} + \mathbf{AS}_l\mathbf{q}_f = \mathbf{r} \tag{44}$$

This equation demonstrates the equivalence of the kinematic descriptions used in the floating frame of reference formulation and the absolute nodal coordinate formulation. Therefore, the coordinate transformation of Eq. 43 can be used to obtain the

nonlinear mass matrix and the inertia shape integrals used in the floating frame of reference formulation from the constant mass matrix used in the absolute nodal coordinate formulation, as demonstrated in the following section using the consistent mass formulation.

7. EQUIVALENCE OF THE INERTIA FORCES

The absolute nodal coordinate formulation leads to a constant mass matrix and zero Coriolis and centrifugal force vector. The floating frame of reference approach, on the other hand leads to a complex highly nonlinear mass matrix and highly nonlinear Coriolis and centrifugal force vectors. Nonetheless, the inertia forces obtained using the two formulations are equivalent as will be demonstrated in this section. In the following section, several applications will be used to compare the results obtained using the two methods.

Exact rigid body motion can be described using the absolute nodal coordinate formulation, only when a consistent mass approach is used. It will be demonstrated in this section that, when consistent mass approach is used, the nonlinear mass matrix and the inertia shape integrals of the floating frame of reference approach can be systematically obtained using the coordinate transformation presented in the preceding section. Equally important, the inertia mass matrix of the rigid body can also be obtained using a similar transformation.

Using the coordinate partitioning of Eq. 32, it can be shown that the mass matrix of the deformable beam element, in the floating frame of reference formulation, can be expressed in terms of the inertia shape integrals of Eq. 35 as [5]

$$\mathbf{M}_f = \begin{bmatrix} m\mathbf{I} & \mathbf{A}_\theta \bar{\mathbf{S}} \mathbf{q}_f & \mathbf{A}\bar{\mathbf{S}} \\ & \mathbf{q}_f^T \mathbf{m}_{ff} \mathbf{q}_f & \mathbf{q}_f^T \tilde{\mathbf{S}} \\ Symmetric & & \mathbf{m}_{ff} \end{bmatrix} \tag{45}$$

Where $\mathbf{I}$ in this equation is a 2×2-identity matrix, m is the mass of the element, and $\mathbf{A}_\theta$ is the partial derivative of the transformation matrix $\mathbf{A}$ with respect to the orientation coordinate θ. The velocity transformation between the coordinates used in the two formulations can be written as

$$\dot{\mathbf{e}} = \mathbf{B}\dot{\mathbf{q}} = \begin{bmatrix} \mathbf{B}_R & \mathbf{B}_\theta & \mathbf{B}_f \end{bmatrix} \begin{bmatrix} \dot{\mathbf{R}} \\ \dot{\theta} \\ \dot{\mathbf{q}}_f \end{bmatrix} \tag{46}$$

where $\mathbf{B}$ is a velocity transformation matrix. Let $\mathbf{M}_a$ be the mass matrix obtained using the absolute nodal coordinate formulation (Eq. 9), the mass matrix $\mathbf{M}_f$ that results from the use of the floating frame of reference formulation can be simply obtained as

$$\mathbf{M}_f = \mathbf{B}^T \mathbf{M}_a \mathbf{B} = \begin{bmatrix} \mathbf{B}_R^T \mathbf{M}_a \mathbf{B}_R & \mathbf{B}_R^T \mathbf{M}_a \mathbf{B}_\theta & \mathbf{B}_R^T \mathbf{M}_a \mathbf{B}_f \\ & \mathbf{B}_\theta^T \mathbf{M}_a \mathbf{B}_\theta & \mathbf{B}_\theta^T \mathbf{M}_a \mathbf{B}_f \\ Symmetric & & \mathbf{B}_f^T \mathbf{M}_a \mathbf{B}_f \end{bmatrix} \tag{47}$$

The elementary shape integrals of Eq. 35 can be determined by comparing Eqs. 45 and 47. The use of this procedure shows that the nonlinear mass matrix and the inertia

shape integrals of the floating frame of reference formulation can be systematically evaluated using the constant mass matrix $\mathbf{M}_a$ and the velocity transformation matrix $\mathbf{B}$ as demonstrated by the following example.

7.1. Cubic Interpolating Polynomials

Using the local shape function of Eq. 39 and the definitions of the constant matrices given by Eq. 35, the inertia shape integrals that appear in the nonlinear mass matrix of the floating frame of reference formulation can be evaluated as

$$\overline{\mathbf{S}} = \int_V \rho \mathbf{S}\, dV = \frac{m}{12}\begin{bmatrix} l & 0 & 6 & -l & 0 \\ 0 & l & 0 & 0 & -l \end{bmatrix}$$

$$\mathbf{M}_{ff} = \int_V \rho \mathbf{S}^T \mathbf{S} dV = m\begin{bmatrix} \frac{l^2}{105} & 0 & \frac{13l}{420} & -\frac{l^2}{140} & 0 \\ 0 & \frac{l^2}{105} & 0 & 0 & -\frac{l^2}{140} \\ \frac{13l}{420} & 0 & \frac{13}{35} & -\frac{11l}{210} & 0 \\ -\frac{l^2}{140} & 0 & -\frac{11l}{210} & \frac{l^2}{105} & 0 \\ 0 & -\frac{l^2}{140} & 0 & 0 & \frac{l^2}{105} \end{bmatrix} \tag{48}$$

$$\tilde{\mathbf{S}} = \int_V \rho \mathbf{S}^T \tilde{\mathbf{I}} \mathbf{S} dV = m\begin{bmatrix} 0 & \frac{l^2}{105} & 0 & 0 & -\frac{l^2}{140} \\ -\frac{l^2}{105} & 0 & -\frac{13l}{420} & \frac{l^2}{140} & 0 \\ 0 & \frac{13l}{420} & 0 & \frac{11l}{210} & -\frac{11l}{210} \\ 0 & -\frac{l^2}{140} & 0 & 0 & \frac{l^2}{105} \\ \frac{l^2}{140} & 0 & \frac{11l}{210} & -\frac{l^2}{105} & 0 \end{bmatrix}$$

Differentiating Eq. 43 with respect to time, one obtains

$$\dot{\mathbf{e}} = \mathbf{B}\dot{\mathbf{q}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -q_1 \sin\theta - q_2 \cos\theta & \cos\theta & -\sin\theta & 0 & 0 & 0 \\ 0 & 0 & q_1 \cos\theta - q_2 \sin\theta & \sin\theta & \cos\theta & 0 & 0 & 0 \\ 1 & 0 & -q_3 \sin\theta & 0 & 0 & \cos\theta & 0 & 0 \\ 0 & 1 & q_3 \cos\theta & 0 & 0 & \sin\theta & 0 & 0 \\ 0 & 0 & -q_4 \sin\theta - q_5 \cos\theta & 0 & 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & q_4 \cos\theta - q_5 \sin\theta & 0 & 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \dot{R}_x \\ \dot{R}_y \\ \dot{\theta} \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \end{bmatrix} \quad (49)$$

which defines the velocity transformation **B** as

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -q_1 \sin\theta - q_2 \cos\theta & \cos\theta & -\sin\theta & 0 & 0 & 0 \\ 0 & 0 & q_1 \cos\theta - q_2 \sin\theta & \sin\theta & \cos\theta & 0 & 0 & 0 \\ 1 & 0 & -q_3 \sin\theta & 0 & 0 & \cos\theta & 0 & 0 \\ 0 & 1 & q_3 \cos\theta & 0 & 0 & \sin\theta & 0 & 0 \\ 0 & 0 & -q_4 \sin\theta - q_5 \cos\theta & 0 & 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & q_4 \cos\theta - q_5 \sin\theta & 0 & 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \quad (50)$$

Using this matrix and Eq. 47, it can be shown that

$$\mathbf{M}_f = \mathbf{B}^T \mathbf{M}_a \mathbf{B} =$$

$$\begin{bmatrix} m\mathbf{I} & \mathbf{A}_\theta \begin{bmatrix} \frac{ml}{12} & 0 & \frac{m}{2} & -\frac{ml}{12} & 0 \\ 0 & \frac{ml}{12} & 0 & 0 & -\frac{ml}{12} \end{bmatrix} \mathbf{q}_f & \mathbf{A} \begin{bmatrix} \frac{ml}{12} & 0 & \frac{m}{2} & -\frac{ml}{12} & 0 \\ 0 & \frac{ml}{12} & 0 & 0 & -\frac{ml}{12} \end{bmatrix} \\ & m\mathbf{q}_f^T \begin{bmatrix} \frac{l^2}{105} & 0 & \frac{13l}{420} & -\frac{l^2}{140} & 0 \\ 0 & \frac{l^2}{105} & 0 & 0 & -\frac{l^2}{140} \\ \frac{13l}{420} & 0 & \frac{13}{35} & -\frac{11l}{210} & 0 \\ -\frac{l^2}{140} & 0 & -\frac{11l}{210} & \frac{l^2}{105} & 0 \\ 0 & -\frac{l^2}{140} & 0 & 0 & \frac{l^2}{105} \end{bmatrix} \mathbf{q}_f & m\mathbf{q}_f^T \begin{bmatrix} 0 & \frac{l^2}{105} & 0 & 0 & -\frac{l^2}{140} \\ -\frac{l^2}{105} & 0 & -\frac{13l}{420} & \frac{l^2}{140} & 0 \\ 0 & \frac{13l}{420} & 0 & \frac{11l}{210} & -\frac{11l}{210} \\ 0 & -\frac{l^2}{140} & 0 & 0 & \frac{l^2}{105} \\ \frac{l^2}{140} & 0 & \frac{11l}{210} & -\frac{l^2}{105} & 0 \end{bmatrix} \\ & \mathit{Symmetric} & m \begin{bmatrix} \frac{l^2}{105} & 0 & \frac{13l}{420} & -\frac{l^2}{140} & 0 \\ 0 & \frac{l^2}{105} & 0 & 0 & -\frac{l^2}{140} \\ \frac{13l}{420} & 0 & \frac{13}{35} & -\frac{11l}{210} & 0 \\ -\frac{l^2}{140} & 0 & -\frac{11l}{210} & \frac{l^2}{105} & 0 \\ 0 & -\frac{l^2}{140} & 0 & 0 & \frac{l^2}{105} \end{bmatrix} \end{bmatrix} \quad (51)$$

Comparing this matrix with Eq. 45, the shape integrals presented in Eq. 48 can be easily identified, demonstrating the equivalence of the inertia forces used in the two formulations. This example also demonstrates that the nonlinear mass matrix and all the inertia shape integrals of the floating frame of reference formulation can be obtained from the constant consistent mass matrix used in linear structural dynamics.

7.2. Rigid Body Inertia

In the case of the consistent mass formulation, the exact rigid body inertia of the beam can be obtained as a special case of the more general development presented in this section. In the case of a rigid body motion, one has

$$q_1 = q_4 = 1, \quad q_3 = l, \quad q_2 = q_5 = 0 \tag{52}$$

In this special case, the transformation of Eq. 49 reduces to

$$\dot{\mathbf{e}} = \mathbf{B}\dot{\mathbf{q}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\sin\theta \\ 0 & 0 & \cos\theta \\ 1 & 0 & -l\sin\theta \\ 0 & 1 & l\cos\theta \\ 0 & 0 & -\sin\theta \\ 0 & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \dot{R}_x \\ \dot{R}_y \\ \dot{\theta} \end{bmatrix} \tag{53}$$

Using the velocity transformation matrix in this equation and the mass matrix $\mathbf{M}_a$ of Eq. 8, it can be shown that in the case of a rigid body motion the mass matrix of the element reduces to

$$\mathbf{M}_f = \mathbf{B}^T\mathbf{M}_a\mathbf{B} = m\begin{bmatrix} 1 & 0 & -\frac{l}{2}\sin\theta \\ 0 & 1 & \frac{l}{2}\cos\theta \\ -\frac{l}{2}\sin\theta & \frac{l}{2}\cos\theta & \frac{l^2}{3} \end{bmatrix} \tag{54}$$

8. APPLICATIONS

In order demonstrate the use of the absolute nodal coordinate formulation in the dynamic simulation of flexible multibody systems, two examples are considered in this section. The results obtained using the absolute nodal coordinate formulation are compared with the results obtained using the floating frame of reference formulation. The two examples considered are the free falling of a flexible pendulum under its own weight, and a flexible slider-crank mechanism driven by a moment applied to the crankshaft. Both the crankshaft and the connecting rod of the slider-crank mechanism are assumed to be flexible bodies. It is important, however, to point out that the floating frame of reference formulation can only be used in the case of small deformation because the deformation of the bodies are expressed in terms of mode shapes. The absolute nodal coordinate formulation, on the other hand, can be used in the small as well as in the large deformation analysis.

8.1. Flexible Pendulum

The first example considered in this section is the free falling of the flexible pendulum shown in Fig. 3. The pendulum, which is horizontal in its initial position, falls under the effect of gravity. The beam has a length of 0.4 m, a cross sectional area of 0.0018 m^2, a second moment of area of 1.215 E-08 m^4, a mass density of 5540 Kg/m^3 and a modulus of elasticity of 1.0 E 09 N/m^2. The beam is divided into 10 elements. In the floating frame of reference formulation, 10 elastic modes are used to describe flexibility of the pendulum rod. The body frame of reference of the flexible pendulum is assumed to

be rigidly attached to its end at the pin joint. Note that in the absolute nodal coordinate formulation 42 degrees of freedom are used, as compared to 13 coordinates in the floating frame of reference formulation, 10 of them describe the elastic deformation.

Figure 4 shows the angular orientation of the flexible pendulum versus time obtained using the two formulations. A very good agreement can be observed between the two methods. Figure 5 shows the transverse displacement of the tip node of the pendulum versus time. The results presented in this figure shows a good agreement between the absolute nodal coordinate formulation and the floating frame of reference formulation.

8.2. Flexible Slider-Crank Mechanism

The second example used in this section to demonstrate the use of the absolute nodal coordinate formulation in the simulation of flexible multibody systems is the flexible slider-crank mechanism shown in Fig. 6. The connecting rod is assumed to be much more flexible than the crankshaft and the slider block is assumed to be rigid and massless. In the initial position, both the connecting rod and crankshaft are assumed to be horizontal. The mechanism is assumed to be driven by a moment applied at the crankshaft. The crankshaft has a length of 0.152 m, a cross sectional area of 7.854E-05 m^2, a second moment of area of 4.909E-10 m^4, a mass density of 2770 Kg/m^3 and a modulus of elasticity of 1.0 E 09 N/m^2. The connecting rod is a beam of length 0.304 m, and has the same cross sectional dimension and material properties as the crankshaft, with the exception of the modulus of elasticity, which is assumed to be 0.5 E 08 N/m^2. In the

dynamic model used in this study, the crankshaft is divided into three finite elements and the connecting rod is divided into eight elements. In the floating frame of reference formulation, three mode shapes are used to describe the flexibility of the crankshaft and five mode shapes are used for the connecting rod.

Two simulation cases were performed. In the first case, the moment applied at the crankshaft is given by

$$M(t)=\left[0.01\left(1-e^{-\frac{t}{0.167s}}\right)\right]Nm \tag{55}$$

In the second case, the moment is assumed to be

$$M(t)=\begin{cases}\left[0.01\left(1-e^{-\frac{t}{0.167s}}\right)\right]Nm & t\le 0.7s\\ 0 & t>0.7s\end{cases} \tag{56}$$

Two coordinates are used to compare the results obtained using the absolute nodal coordinate formulation and the floating frame of reference formulation. These are the X position of the slider block and the transverse deformation of the middle point of the connecting rod. Figures 7 and 8 show the slider block position in the two cases of the applied moments. These two figures show good agreement between the results obtained using the absolute nodal coordinate formulation and the floating frame of reference approach. Figures 9 and 10 show the transverse deformation of the mid point of the connecting rod. In the first case of the applied moment, when the velocity of the system increases as well as the inertia forces, the deformation becomes relatively large, and

differences between the solutions obtained using the two formulations can be observed. In the second case of the applied moment the transverse deformation remains relatively small. In this case, excellent agreement between the two formulations can be observed, as shown in Fig 10. The only difference is that the solution obtained using the absolute nodal formulation has high frequency signals as the result of including more degrees of freedom as compared to the floating frame of reference formulation.

9. SUMMARY AND CONCLUSIONS

The concerns regarding the use of the classical finite element formulation in the large deformation and rotation analysis of flexible multibody systems are attributed to two main reasons. First, in the classical finite element literature, infinitesimal rotations are used as nodal coordinates in the case of beams and plates. Such a use of coordinates does not lead to the exact representation of a simple rigid body motion as recently demonstrated. Secondly, lumped mass techniques are used in many finite element formulations and computer programs to describe the inertia of deformable bodies. In this paper, the effect of using the consistent mass formulation on the structure of the nonlinear dynamic equations of deformable bodies that undergo large reference displacements is examined. To this end, the absolute nodal coordinate formulation, which can be efficiently used in the large rotations and deformations of deformable bodies that undergo arbitrary displacements, is utilized. In this formulation, no infinitesimal or finite rotations are used as nodal coordinates, instead the slopes and the displacements at the nodal points

are used as element nodal coordinates. Crucial to the success of using this new formulation, however, is the use of a consistent mass approach. This is a necessary requirement, which guarantees that exact modeling of the rigid body inertia can be obtained when the structures rotate as rigid bodies. In this paper, the equivalence of the absolute nodal coordinate formulation and the floating frame of reference formulation which is widely used in flexible multibody simulations is further utilized in order to compare analytically and numerically two different procedures which can be efficiently used in flexible multibody simulations.

In the absolute nodal coordinate formulation, a new interpretation for the nodal coordinates is used. By using this new interpretation of the coordinates, a constant mass matrix can be obtained and as a result the Coriolis and centrifugal forces are equal to zero. The elastic forces, on the other hand, are highly nonlinear functions of the element coordinates. Therefore, little is to be gained by using the small deformation assumptions. The absolute nodal coordinate formulation can be effectively used in the large deformation problems as well as flexible multibody applications as demonstrated in this paper. In addition to the constant simple mass matrix that appears in this formulation, the formulation of some of the joint constraints as well as forces can be very simple as compared to the floating frame of reference approach. Because of the nature of the coordinates used in the floating frame of reference formulation, such a method has been only used in the small deformation analysis of flexible multibody systems. The absolute nodal coordinate formulation does not suffer from this limitation and can be used in the small and large deformation analysis of flexible multibody systems. The applications

used in this paper to compare the results obtained using the absolute nodal coordinate formulation and the results obtained using the floating frame of reference approach show excellent agreement between the two methods in the analysis of small deformations. Discrepancies can be observed between the results obtained using the two methods as the deformation increases.

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APPENDIX

STIFFNESS SHAPE INTEGRALS

The definition of the matrices that appear in the elastic forces can be made simpler if the nodal coordinates are rearranged as:

$$\mathbf{e} = \begin{bmatrix} e_1 & e_3 & e_5 & e_7 & e_2 & e_4 & e_6 & e_8 \end{bmatrix}^T = \begin{bmatrix} \mathbf{e}_x & \mathbf{e}_y \end{bmatrix}^T \tag{57}$$

Define the matrix **A** and **B** as follows:

$$\mathbf{A} = Ea\begin{bmatrix} \frac{6}{5l} & \frac{1}{10} & -\frac{6}{5l} & \frac{1}{10} \\ \frac{1}{10} & \frac{2l}{15} & -\frac{1}{10} & -\frac{l}{30} \\ -\frac{6}{5l} & -\frac{1}{10} & \frac{6}{5l} & -\frac{1}{10} \\ \frac{1}{10} & -\frac{l}{30} & -\frac{1}{10} & \frac{2l}{15} \end{bmatrix}; \quad \mathbf{B} = EI\begin{bmatrix} \frac{12}{l^3} & \frac{6}{l^2} & -\frac{12}{l^3} & \frac{6}{l^2} \\ \frac{6}{l^2} & \frac{4}{l} & -\frac{6}{l^2} & \frac{2}{l} \\ -\frac{12}{l^3} & -\frac{6}{l^2} & \frac{12}{l^3} & -\frac{6}{l^2} \\ \frac{6}{l^2} & \frac{2}{l} & -\frac{6}{l^2} & \frac{4}{l} \end{bmatrix} \tag{58}$$

These matrices can be considered as the axial and bending stiffness matrices that appear in linear structural dynamics. By using the arrangement defined in Eq. 57 and the matrices in Eq. 58, the stiffness shape integrals that appear in the expression of the elastic forces are

$$\begin{aligned} \mathbf{A}_{11} &= \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}; \quad \mathbf{A}_{22} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} \end{bmatrix}; \quad \mathbf{A}_{12} = \begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A} & \mathbf{0} \end{bmatrix}; \\ \mathbf{B}_{11} &= \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}; \quad \mathbf{B}_{22} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix}; \quad \mathbf{B}_{12} = \begin{bmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \end{aligned} \tag{59}$$

and:

$$\begin{aligned} \mathbf{A}_1 &= \begin{bmatrix} -Ea & Ea & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T; \\ \mathbf{A}_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & -Ea & Ea & 0 & 0 \end{bmatrix}^T \end{aligned} \tag{60}$$

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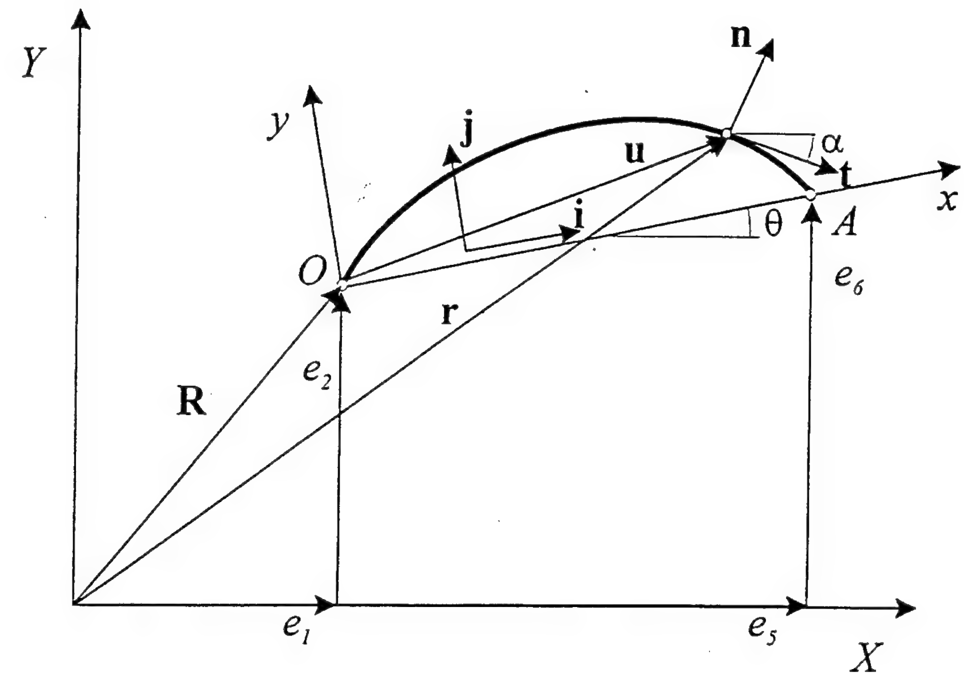

Figure 1. Planar Beam Element

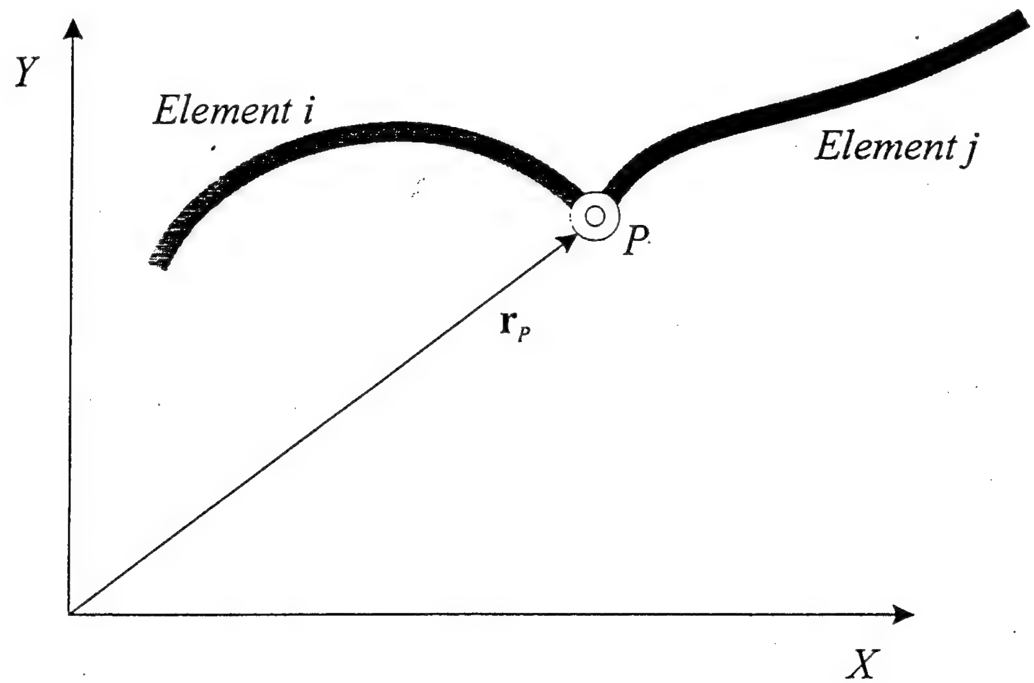

Figure 2. Revolute (Pin) Joint Between Two Elements

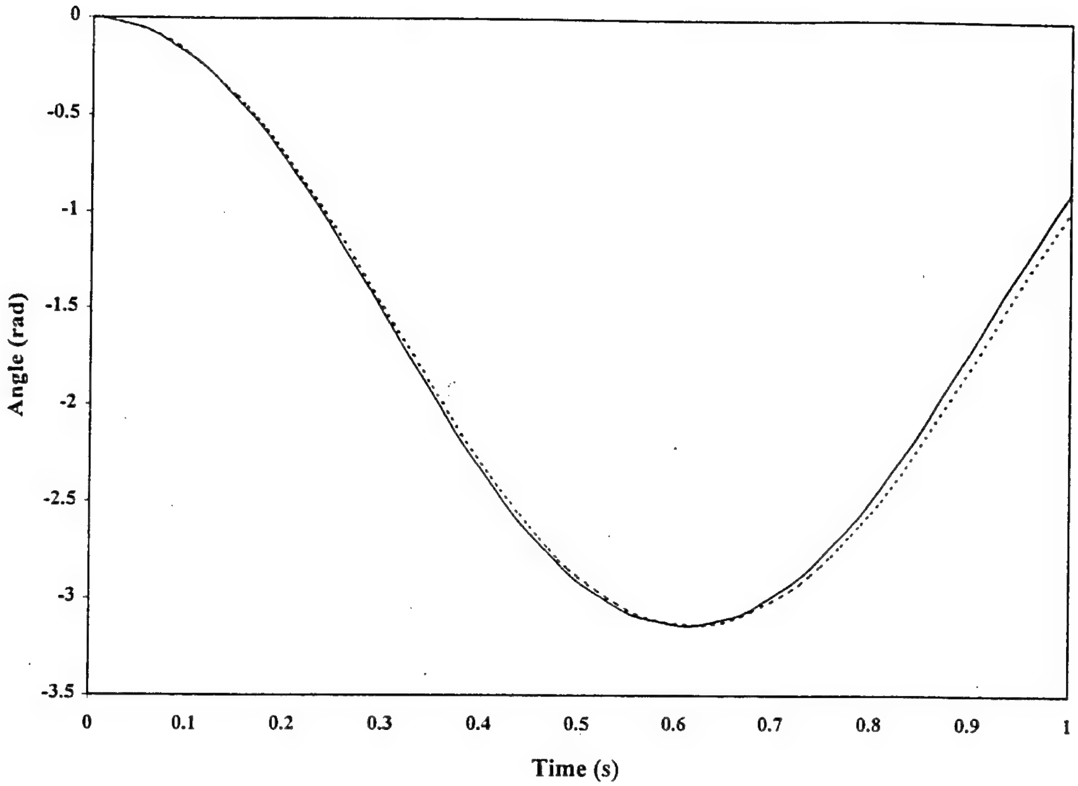

Figure 4. Angular Orientation of the Pendulum.
Absolute Nodal Coordinate Formulaiton: '—',
Floating Frame of Reference Formulation: '…'

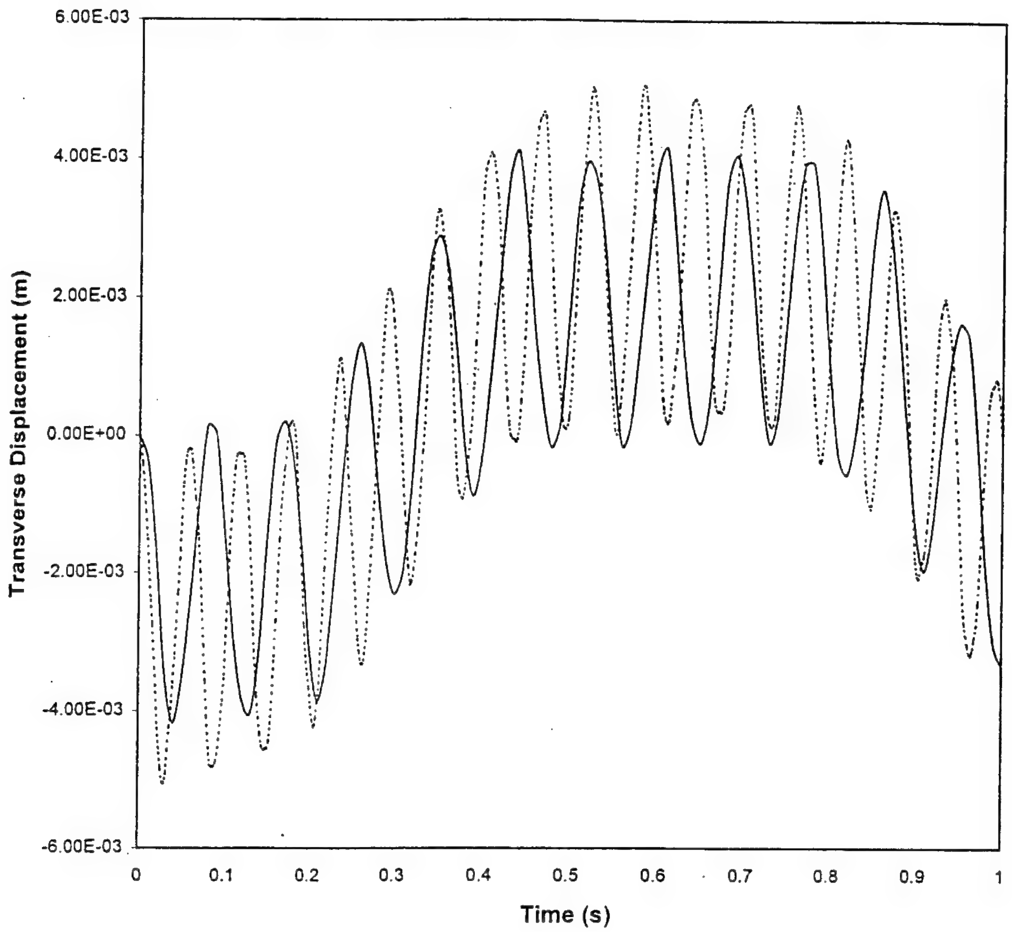

Figure 5. Transverse Deformation of the Tip Point of the Pendulum.
Absolute Nodal Coordiante Formulation: '—',
Floating Frame of Reference Formulation: '…'

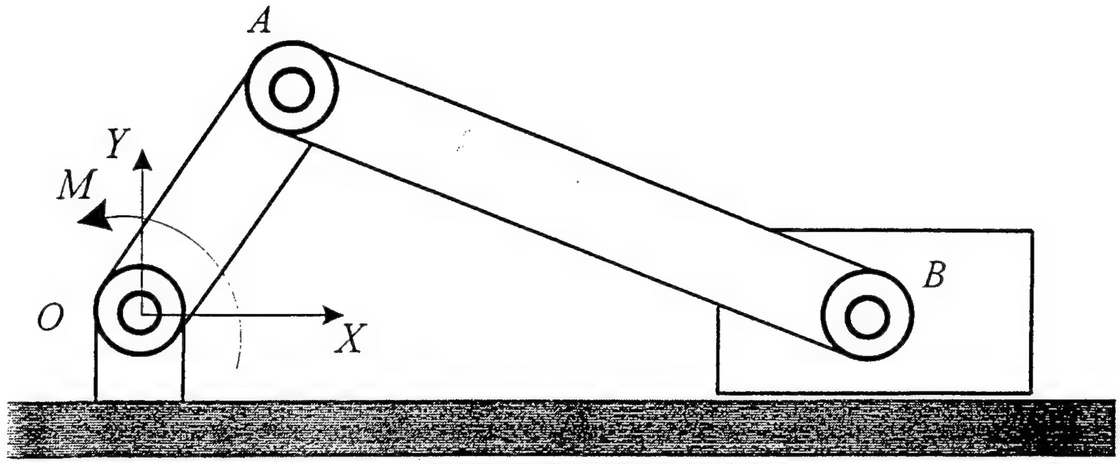

Figure 6. Slider Crank Mechanism.

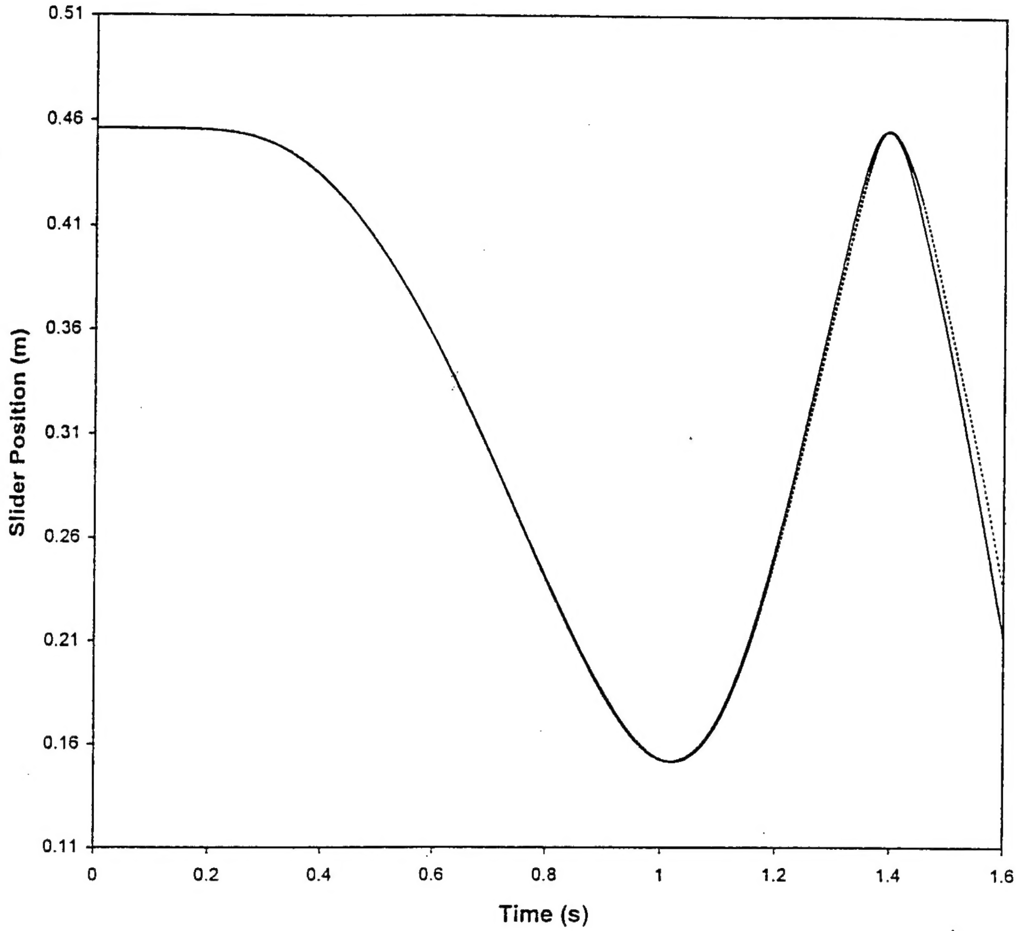

Figure 7. Coordinate of the Slider Block (Moment Defined by Eq. 55).
Abslolute Nodal Coordinate Formulation: '—',
Floating Frame of Reference Formulation: '...'

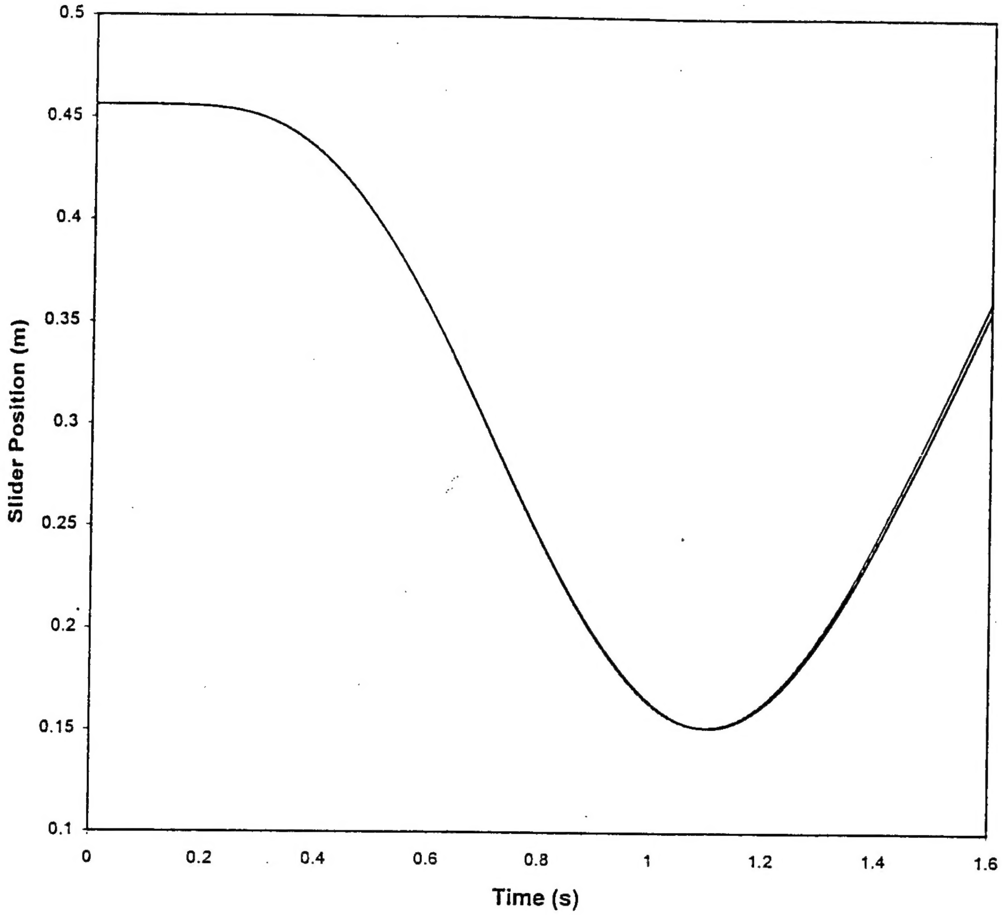

Figure 8.Coordinate of the Slider Block (Moment Defined by Eq. 56).
Abslolute Nodal Coordinate Formulation: '—',
Floating Frame of Reference Formulation: '...'

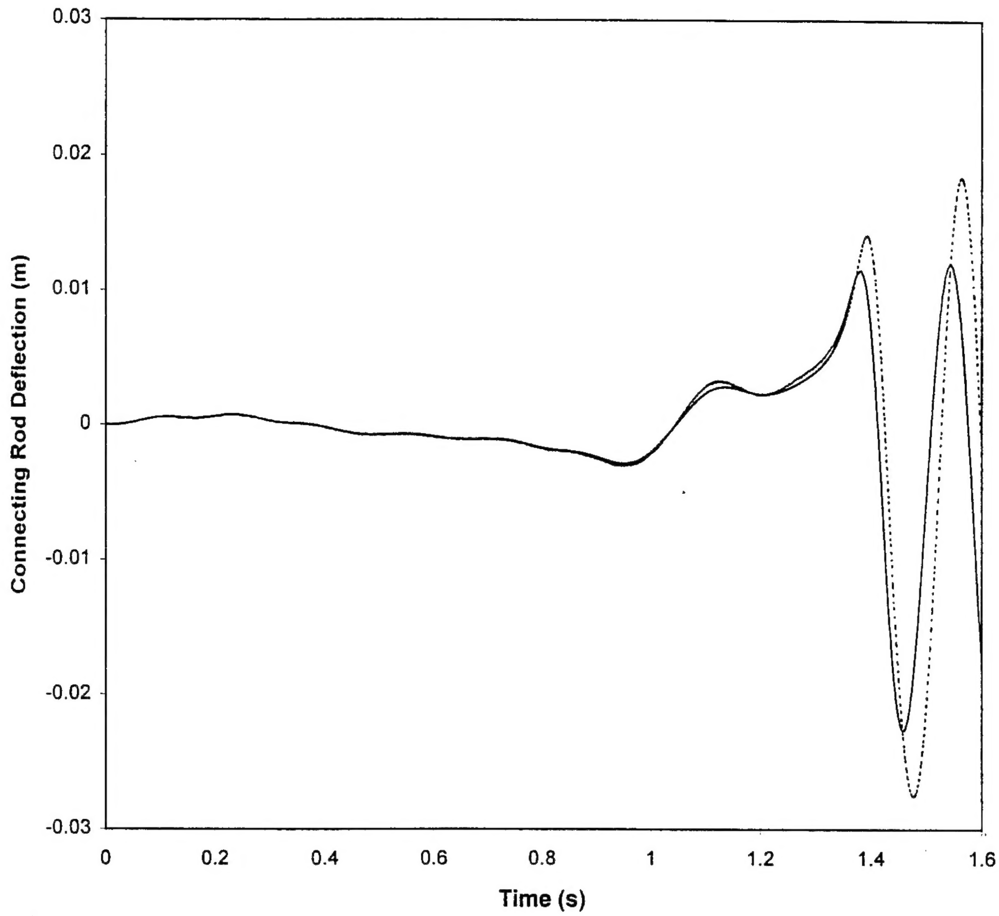

Figure 9. Transverse Deformation of the Mid Point of the Connecting Rod
(Moment Defined by Eq. 55)
Abslolute Nodal Coordinate Formulation: '—',
Floating Frame of Reference Formulation: '…'

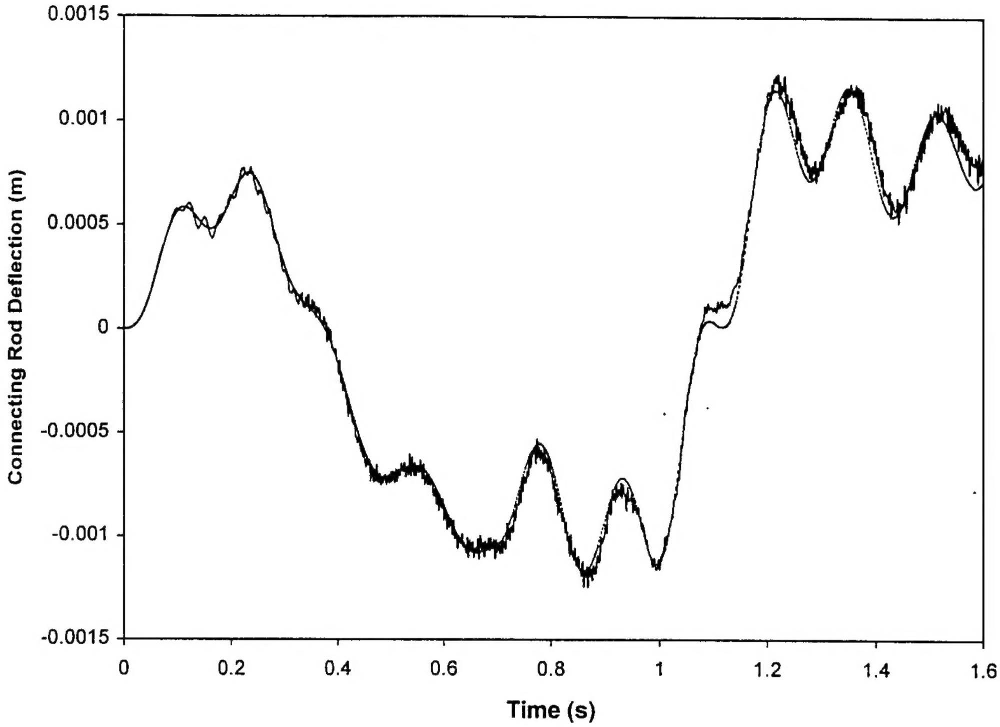

Figure 10. Transverse Deformation of the Mid Point of the Connecting Rod
(Moment Defined by Eq. 56)
Abslolute Nodal Coordinate Formulation: '—',
Floating Frame of Reference Formulation: '…'